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COMMENT

A new two-mode fermion coherent state representation

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Abstract. A technique of integration within the ordered product for a fermion system is introduced. In terms of the technique, a new two-mode fermion coherent state representation is found. Some properties and applications of the new representation are derived.

1. Introduction

It is well known in quantum mechanics [1] that if two operators A and B commute, and if $|\psi\rangle$ is an eigenvector of A, then $B|\psi\rangle$ is also an eigenvector of A, with the same eigenvalue. An interesting question thus arises: if two operators anticommute, can they have a set of eigenvectors in common? To our knowledge, this question has not been paid enough attention in the literature. In this comment, we shall answer this question in terms of the technique of integration within the ordered product (IWOP) for fermion systems, which was put forward in [2]. According to the technique, any two Fermi operators anticommute with each other within : : (the normal product) while any two Grassmann number-Fermi operator pairs commute with each other within : :, and the completeness relation of fermion coherent states can be recast into the normally ordered form, e.g.

$$\int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i |\alpha_i\rangle\langle\alpha_i| = \int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i :\exp[-(\bar{\alpha}_i - a_i^\dagger)(\alpha_i - a_i)]: = 1 \tag{1.1}$$

where $a_i^\dagger$ (a_i) are Fermi creation (annihilation) operators, satisfying

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad a_i^2 = 0 \qquad a_i^{\dagger 2} = 0 \tag{1.2}$$

α_i are Grassmann numbers [3-5]

$$|\alpha_i\rangle = \exp[-\tfrac{1}{2}\bar{\alpha}_i\alpha_i + a_i^\dagger\alpha_i]|0_i\rangle \qquad a_i|\alpha_i\rangle = |\alpha_i\rangle\alpha_i \qquad a_i|0\rangle_i = 0 \tag{1.3}$$

and the following formulae are used:

$$\int \mathrm{d}\alpha_i = 0 \qquad \int \mathrm{d}\alpha_i\alpha_i = 1 \tag{1.4}$$

$$\int \prod_1^N \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i \exp\left[-\sum_{i,j} \bar{\alpha}_i\Lambda_{ij}\alpha_j + \sum_i (\bar{\alpha}_i\eta_i + \bar{\eta}_i\alpha_i)\right] = \det\Lambda \exp\left[\sum_{i,j} \bar{\eta}_i(\Lambda^{-1})_{ij}\eta_j\right] \tag{1.5}$$

$$|0\rangle_{ii}\langle 0| = :\exp[-a_i^\dagger a_i]:. \tag{1.6}$$

We shall show, in section 2, that this technique can provide a direct approach for constructing a new fermion coherent state which is a common eigenstate of the two anticommuting Fermi operators $a_1 + a_2^\dagger$ and $a_1^\dagger - a_2$; some properties and applications of the new state are derived in sections 3 and 4, respectively.

2. New two-mode fermion coherent state

Enlightened by the normally ordered form of the completeness relation (1.1), by virtue of the IWOP technique we can easily prove the following

$$\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi\,:\exp\{-[\bar{\xi}-(a_1^\dagger-a_2)][\xi-(a_1+a_2^\dagger)]\}:=1 \tag{2.1}$$

where $\bar{\xi}$ and ξ are Grassmann numbers, independent of each other. Using (1.6), it is easily seen that (2.1) is equivalent to

$$\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi|\xi\rangle\langle\langle\xi|=1 \tag{2.2}$$

where

$$|\xi\rangle=\exp(-\tfrac{1}{2}\bar{\xi}\xi+a_1^\dagger\xi+\bar{\xi}a_2^\dagger+a_2^\dagger a_1^\dagger)|00\rangle \tag{2.3}$$

$$\langle\langle\xi|=\langle 00|\exp(-\tfrac{1}{2}\bar{\xi}\xi+\bar{\xi}a_1-a_2\xi-a_1a_2). \tag{2.4}$$

Operating with a_1 and $a_2^\dagger$ on $|\xi\rangle$, respectively, we have

$$a_1|\xi\rangle=(\xi-a_2^\dagger-\tfrac{1}{2}\bar{\xi}\xi a_2^\dagger)|00\rangle \tag{2.5}$$

$$a_2^\dagger|\xi\rangle=a_2^\dagger(1-\tfrac{1}{2}\bar{\xi}\xi+a_1^\dagger\xi)|00\rangle. \tag{2.6}$$

On the other hand, we have

$$\xi|\xi\rangle=\xi(1+\bar{\xi}a_2^\dagger+a_2^\dagger a_1^\dagger)|00\rangle. \tag{2.7}$$

As a consequence of (2.5)-(2.7), we obtain the eigenvector equation

$$(a_1+a_2^\dagger)|\xi\rangle=\xi|\xi\rangle. \tag{2.8}$$

Further, operating a_2 and $a_1^\dagger$ on $|\xi\rangle$ respectively, gives us

$$a_2|\xi\rangle=(a_1^\dagger-\bar{\xi}+\tfrac{1}{2}\bar{\xi}\xi a_1^\dagger)|00\rangle \tag{2.9}$$

$$a_1^\dagger|\xi\rangle=a_1^\dagger(1-\tfrac{1}{2}\bar{\xi}\xi+\bar{\xi}a_2^\dagger)|00\rangle. \tag{2.10}$$

Comparing the following equation

$$\bar{\xi}|\xi\rangle=\bar{\xi}(1+a_1^\dagger\xi+a_2^\dagger a_1^\dagger)|00\rangle \tag{2.11}$$

we get another eigenvector equation

$$(a_1^\dagger-a_2)|\xi\rangle=\bar{\xi}|\xi\rangle. \tag{2.12}$$

Owing to (1.2), it is easily seen that

$$\{a_1^\dagger-a_2,\,a_1+a_2^\dagger\}=0 \qquad \left\{\frac{a_1^\dagger-a_2}{\sqrt{2}},\frac{a_1-a_2^\dagger}{\sqrt{2}}\right\}=1 \qquad \left\{\frac{a_1+a_2^\dagger}{\sqrt{2}},\frac{a_1^\dagger+a_2}{\sqrt{2}}\right\}=1. \tag{2.13}$$

We thus reach the conclusion that the two anticommuting operators $a_1+a_2^\dagger$ and $a_1^\dagger-a_2$ have the eigenvector in common. (Note that the two-mode fermion coherent state $|\alpha_1\alpha_2\rangle$ is a common eigenvector of a_1 and a_2; however this is a trivial case.) At the end of this section we point out that although the following integration seems a likely more natural generalisation of (1.1) than (2.1) and also gives:

$$\int \mathrm{d}\bar{\xi}\,\mathrm{d}\xi\,:\exp\{-[\bar{\xi}-(a_1^\dagger+a_2)][\xi-(a_1+a_2^\dagger)]\}:=1 \tag{2.14}$$

it cannot lead us to proceed far, because in the exponential of (2.14) there appears the term $a_2^\dagger a_2$ (not $-\alpha_2^\dagger a_2$!) and we cannot use $:\exp(-a_2^\dagger a_2): = |0\rangle_{22}\langle 0|$ to decompose the exponential of (2.14) into the suitable bra and ket state as we did for (2.1).

3. Some properties of the new state

It is worth pointing out that if we define the adjoint of $|\xi\rangle$ by

$$\langle \xi| = \langle 00|\exp(-\tfrac{1}{2}\bar{\xi}\xi + \bar{\xi}a_1 + a_2\xi + a_1 a_2) \tag{3.1}$$

which is a left eigenvector of $a_1^\dagger + a_2$ with eigenvalue $\bar{\xi}$, we see that $a_1^\dagger + a_2$ does not anticommute with $a_1 + a_2^\dagger$. Moreover, if we carry out the integration $\int d\bar{\xi}\, d\xi |\xi\rangle\langle\xi|$ in terms of the IWOP technique, we do not get 1. Instead, we have

$$\begin{aligned}\int d\bar{\xi}\, d\xi |\xi\rangle\langle\xi| &= \int d\bar{\xi}\, d\xi \colon \exp(-\bar{\xi}\xi + (a_1^\dagger + a_2)\xi + \bar{\xi}(a_2^\dagger + a_1) + a_2^\dagger a_1^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2)\colon \\ &= \colon\exp(-2a_2^\dagger a_2)\colon.\end{aligned} \tag{3.2}$$

Using

$$\begin{aligned} &|0\rangle_{22}\langle 0| = a_2 a_2^\dagger \qquad |1\rangle_{22}\langle 1| = a_2^\dagger a_2 \\ &\colon\exp(-2a_2^\dagger a_2)\colon = 1 - 2a_2^\dagger a_2 = |0\rangle_{22}\langle 0| - |1\rangle_{22}\langle 1| = (-1)^{N_2} \qquad N_2 \equiv a_2^\dagger a_2 \end{aligned} \tag{3.3}$$

we can put (3.2) into the form

$$\int d\bar{\xi}\, d\xi |\xi\rangle\langle\xi|(-1)^{N_2} = 1 \qquad \text{or} \qquad \int d\bar{\xi}\, d\xi (-1)^{N_2}|\xi\rangle\langle\xi| = \int d\bar{\xi}\, d\xi |\xi\rangle\!\rangle\langle\xi| = 1 \tag{3.4}$$

which is in agreement with (2.2) since

$$(-1)^{N_2}a_2(-1)^{N_2} = -a_2 \qquad (-1)^{N_2}|\xi\rangle = \exp(-\tfrac{1}{2}\bar{\xi}\xi + a_1^\dagger\xi - \bar{\xi}a_2^\dagger - a_2^\dagger a_1^\dagger)|00\rangle = |\xi\rangle\!\rangle. \tag{3.5}$$

As a result of (2.8), (2.12) and (3.5), we have

$$\langle\!\langle \xi|(a_1^\dagger - a_2) = \bar{\xi}\langle\!\langle \xi| \tag{3.6}$$

$$\langle\!\langle \xi|(a_1 + a_2^\dagger) = \xi\langle\!\langle \xi| \tag{3.7}$$

which states that $\langle\!\langle \xi|$ is a left eigenvector of $a_1^\dagger - a_2$ and $a_1 + a_2^\dagger$ simultaneously. Combining (2.8), (2.12), (3.6) and (3.7) together, we obtain

$$\langle\!\langle \xi'|(a_1 + a_2^\dagger)|\xi\rangle = \xi\langle\!\langle \xi'|\xi\rangle = \xi'\langle\!\langle \xi'|\xi\rangle \tag{3.8}$$

$$\langle\!\langle \xi'|(a_1^\dagger - a_2)|\xi\rangle = \bar{\xi}\langle\!\langle \xi'|\xi\rangle = \bar{\xi}'\langle\!\langle \xi'|\xi\rangle. \tag{3.9}$$

The δ function of the Grassmann algebra can be defined as

$$\delta(\xi - \xi') = \xi - \xi' \qquad \delta(\bar{\xi} - \bar{\xi}') = \bar{\xi} - \bar{\xi}' \tag{3.10}$$

so (3.8) and (3.9) become

$$\langle\!\langle \xi'|\xi\rangle = \delta(\xi - \xi')\delta(\bar{\xi} - \bar{\xi}') = \langle \xi|\xi'\rangle\!\rangle \tag{3.11}$$

which embodies the orthogonality and is consistent with the completeness relation (2.2). Hence $|\xi\rangle$ and $\langle\!\langle \xi|$ make up a new representation, even though $|\xi\rangle$ and $\langle\!\langle \xi|$ are not 'Hermitian conjugate' to each other. It is due to the IWOP technique that we are able to find such a case easily.

4. Application of the new representation

As an application of the new representation, let us consider the following integration:

$$U \equiv \int d\bar{\xi}\, d\xi \left| \begin{pmatrix} s^* & -r \\ r^* & s \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \right\rangle\!\!\Big\rangle \Big\langle\!\!\Big\langle \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \right| \qquad |s|^2+|r|^2=1 \tag{4.1}$$

where we have rewritten $\langle\xi|$ as $\langle\langle(\begin{smallmatrix}\xi\\ \bar{\xi}\end{smallmatrix})|$ and

$$|\xi\rangle\!\rangle \equiv |\xi, \bar{\xi}\rangle\!\rangle \equiv \left| \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \right\rangle\!\!\Big\rangle \tag{4.2}$$

$$\left| \begin{pmatrix} s^* & -r \\ r^* & s \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \right\rangle\!\!\Big\rangle = \exp\left(-\frac{\bar{\xi}\xi}{2} + a_1^\dagger (s^*\xi - r\bar{\xi}) - (r^*\xi + s\bar{\xi}) a_2^\dagger - a_2^\dagger a_1^\dagger \right) |00\rangle. \tag{4.3}$$

Substituting (4.3) into (4.2) and performing the integration in the light of the IWOP technique, we obtain

$$\begin{aligned} U = \int d\bar{\xi}\, d\xi \colon & \exp[-\bar{\xi}\xi + (a_1^\dagger s^* + r^* a_2^\dagger + a_2)\xi + \bar{\xi}(r a_1^\dagger - s a_2^\dagger + a_1) \\ & - a_2^\dagger a_1^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2] \colon \\ & = \colon \exp[(s^* - 1) a_1^\dagger a_1 + (s-1) a_2^\dagger a_2 + r^* a_2^\dagger a_1 - r a_1^\dagger a_2] \colon \\ & = \colon \exp\left\{ (a_1^\dagger a_2^\dagger) \left[\begin{pmatrix} s^* & -r \\ r^* & s \end{pmatrix} - \mathbb{1} \right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\} \colon \qquad \mathbb{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.4}$$

which shows the mapping of the classical transformation to a qunatum operator. Using (4.1), (3.11) and

$$(s a_2^\dagger - r a_1^\dagger)(a_1^\dagger s^* + r^* a_2^\dagger) = a_2^\dagger a_1^\dagger$$

we have

$$\begin{aligned} U|\xi\rangle\!\rangle = & \int d\bar{\xi}'\, d\xi \left| \begin{pmatrix} s^* & -r \\ r^* & s \end{pmatrix} \begin{pmatrix} \xi' \\ \bar{\xi}' \end{pmatrix} \right\rangle\!\!\Big\rangle \delta(\xi - \xi')\delta(\bar{\xi} - \bar{\xi}') \\ & = \exp\left(-\frac{\bar{\xi}\xi}{2} + (a_1^\dagger s^* + r^* a_2^\dagger)\xi - \bar{\xi}(s a_2^\dagger - r a_1^\dagger) - (s a_2^\dagger - r a_1^\dagger)(a_1^\dagger s^* + r^* a_2^\dagger) \right) |00\rangle. \end{aligned} \tag{4.5}$$

It is not difficult to prove that U is unitary, thus from (4.5), (3.5) and $U|00\rangle = |00\rangle$, which is obtained from (4.4), we have

$$b_1^\dagger \equiv U a_1^\dagger U^{-1} = a_1^\dagger s^* + r^* a_2^\dagger \qquad b_2^\dagger \equiv U a_2^\dagger U^{-1} = s a_2^\dagger - r a_1^\dagger. \tag{4.6}$$

This transformation keeps the anticommutative relation invariant, e.g. $\{b_i, b_j^\dagger\} = \delta_{ij}$, $b_i^2 = 0$. As another application, we examine the following integral-form operator

$$V \equiv \int d\bar{\eta}\, d\eta \int d\bar{\xi}\, d\xi \left| R \begin{pmatrix} \xi \\ \bar{\xi} \\ \eta \\ \bar{\eta} \end{pmatrix} \right\rangle\!\!\Big\rangle\!\Big\rangle \Big\langle\!\Big\langle\!\!\Big\langle \begin{pmatrix} \xi \\ \bar{\xi} \\ \eta \\ \bar{\eta} \end{pmatrix} \right| \qquad R \equiv \begin{pmatrix} s & 0 & 0 & -r^* \\ 0 & s^* & -r & 0 \\ 0 & r^* & s & 0 \\ r & 0 & 0 & s^* \end{pmatrix} \tag{4.7}$$

where

$$\left| \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} \right\rangle\!\!\Big\rangle \equiv |\eta\rangle\!\rangle \equiv |\eta, \bar{\eta}\rangle\!\rangle = \exp\left(-\frac{\bar{\eta}\eta}{2} + a_3^\dagger \eta - \bar{\eta} a_4^\dagger - a_4^\dagger a_3^\dagger \right) \| 00\rangle \tag{4.8}$$

is another state similar to $|\xi\rangle$ but with $a_3^\dagger$, $a_4^\dagger$ being another set of two-mode Fermi creation operators, and $\|00\rangle$ is the ground state eliminated by a_3 and a_4; thus we have

$$\|00\rangle\langle 00\| = :\exp(-a_3^\dagger a_3 - a_4^\dagger a_4):. \tag{4.9}$$

In terms of the IWOP technique, we perform the integration in (4.7) to obtain

$$\begin{aligned}
V = \int d\bar\eta\, d\eta \int & d\bar\xi\, d\xi \exp[-\tfrac{1}{2}(s^*\bar\xi - r\eta)(s\xi - r^*\bar\eta) + a_1^\dagger(s\xi - r^*\bar\eta) \\
& -(s^*\bar\xi - r\eta)a_2^\dagger - a_2^\dagger a_1^\dagger - \tfrac{1}{2}(r\xi + s^*\bar\eta)(r^*\bar\xi + s\eta) \\
& + a_3^\dagger(r^*\bar\xi + s\eta) - (r\xi + s^*\bar\eta)a_4^\dagger - a_4^\dagger a_3^\dagger]|00\rangle\|00\rangle\langle 00\|\langle 00| \\
& \times \exp(-\tfrac{1}{2}\bar\xi\xi + \bar\xi a_1 + a_2\xi + a_1 a_2 - \tfrac{1}{2}\bar\eta\eta + \bar\eta a_3 + a_4\eta + a_3 a_4) \\
= \int & d\bar\eta\, d\eta\, d\bar\xi\, d\xi :\exp[-|s|^2(\bar\xi\xi + \bar\eta\eta) - rs\xi\eta - s^* r^*\bar\eta\bar\xi + a_1^\dagger(s\xi - r^*\bar\eta) \\
& -(s^*\bar\xi - r\eta)a_2^\dagger + a_3^\dagger(r^*\bar\xi + s\eta) - (r\xi + s^*\bar\eta)a_4^\dagger + \bar\xi a_1 + a_2\xi + \bar\eta a_3 \\
& + a_4\eta - (a_1^\dagger + a_2)(a_1 - a_2^\dagger) - (a_3^\dagger + a_4)(a_3 - a_4^\dagger)]: \\
= |s|^2 & \exp[(r/s)a_2^\dagger a_4^\dagger - (r^*/s^*)a_1^\dagger a_3^\dagger] :\exp[((1/s)-1)(a_4^\dagger a_4 + a_2^\dagger a_2) \\
& + ((1/s^*)-1)(a_3^\dagger a_3 + a_1^\dagger a_1): \exp[(r^*/s)a_2 a_4 - (r/s^*)a_1 a_3].
\end{aligned} \tag{4.10}$$

By virtue of the operator identity

$$\exp(\lambda a_i^\dagger a_i) = 1 + a_i^\dagger a_i(e^\lambda - 1) = :\exp[(e^\lambda - 1)a_i^\dagger a_i]: \tag{4.11}$$

we can put (4.10) into the form

$$V = V'V'' \tag{4.12}$$

$$V' = \exp\left(-\frac{r}{s}\, a_4^\dagger a_2^\dagger\right) \exp\left[(a_4^\dagger a_4 + a_2^\dagger a_2 - 1)\ln\left(\frac{1}{s}\right)\right] \exp\left(\frac{r^*}{s}\, a_2 a_4\right) \tag{4.13}$$

$$V'' = \exp\left(-\frac{r^*}{s^*}\, a_1^\dagger a_3^\dagger\right) \exp\left[(a_3^\dagger a_3 + a_1^\dagger a_1 - 1)\ln\left(\frac{1}{s^*}\right)\right] \exp\left(\frac{r}{s^*}\, a_3 a_1\right) \tag{4.14}$$

which can generate a fermionic Bogolyubov–Valatin transformation [6, 7]

$$(V')^{-1}a_4 V' = s^* a_4 - r a_2^\dagger \qquad (V')^{-1}a_2 V' = s^* a_2 + r a_4^\dagger \tag{4.15}$$

$$(V'')^{-1}a_3 V'' = s a_3 + r^* a_1^\dagger \qquad (V'')^{-1}a_1 V'' = s a_1 - r^* a_3^\dagger. \tag{4.16}$$

Therefore, from (4.7) and (4.10) we see that the R transformation in Grassmann number space maps into the tensor product of fermionic Boglyubov–Valatin unitary operators in Hilbert space. Further, using (4.7) and (3.11) we have

$$V\left|\begin{pmatrix}\xi\\ \bar\xi\\ \eta\\ \bar\eta\end{pmatrix}\right\rangle\!\!\Bigg\rangle\!\!\Bigg\rangle = \left|R\begin{pmatrix}\xi\\ \bar\xi\\ \eta\\ \bar\eta\end{pmatrix}\right\rangle\!\!\Bigg\rangle\!\!\Bigg\rangle.$$

To summarise, the IWOP technique for fermionic systems can help us not only to derive the N-fermion permutation operator [2], but also to deduce the new two-mode fermion coherent state with which some physical applications are obtained, as shown in (4.1) and (4.7). This kind of state satisfies its orthonormal and complete relations manifestly in (3.11) and (2.2), and provides us with a non-trivial example of the fact that two anticommuting operators can possess a set of eigenvectors in common. Combining the results obtained in [2] and in this comment, we can conclude that the IWOP technique makes the fermion coherent state much more useful.

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